Periodic Neural Activity Induced by Network Complexity

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(Dated: February 6, 2008)

We study a model for neural activity on the small-world topology of Watts and Strogatz and on the scale-free topology of Barabási and Albert. We find that the topology of the network connections may spontaneously induce periodic neural activity, contrasting with chaotic neural activities exhibited by regular topologies. Periodic activity exists only for relatively small networks and occurs with higher probability when the rewiring probability is larger. The average length of the periods increases with the square root of the network size.

PACS numbers: 05.45.Xt, 05.50.+q, 89.75.Da, 89.75.Fb

The human brain is the most fascinating processor, consisting of about ten billion neurons. These neurons are connected to each other by synapses, forming together the neural network. The synapses transmit stimuli through different concentrations of Na^+ and K^+ ions. The neurons communicate with each other through electrical impulses. Each time a neuron is charged beyond a certain threshold by the connected neurons, it "fires" an electrical discharge through its axon which through synapses transmits charges to the dendrites of other neurons. While most synapses just connect close by neurons, a few of them also can be long range and connect to a neuron in a distant area of the brain. These few far-reaching connections seem to be crucial for the coherent functioning of the brain. Such a mixed network structure of many short- and a few long-range connections is the trademark of small-world networks as introduced in a seminal work by Watts and Strogatz [1, 2, 3]. In vitro studies of neuronal networks have in fact been grown and analyzed and found to have small-world properties [4]. Another direction of research has been analyzing avalanches of neurons firing, reporting that there exists some criticality or scale-free behavior. This has been observed experimentally in organotypic cultures from coronal slices of rat cortex [5] and been modeled as a self-organized critical process [6].

In brain research, the appearance of periodic cycles of firing sequences is commonly observed, being considered as responsible for the origin of various body clocks or even been interpreted as the realization of some simple thoughts. One of the fundamental questions is how does such a seemingly disordered system as the brain synchronizes the activity as to produce these periodic signals. It is the aim of the present work to present a simple neural model on Watts-Strogatz and Barabási-Albert networks and show that it can, under suitable circumstances, spontaneously generate periodic time series in its activity. Similar studies have already been performed using the Hodgkin-Huxley model [7, 8, 9] where it was found that the Watts-Strogatz network has fast coherent oscillations as opposed to other types of graphs. Also integrate-and-fire neurons have been studied on small-world networks and a transition between self-sustained persistent activity and failure has been reported [10]. Other properties like, for instance, the background of the neural activity, seem rather unaffected by the small world properties as observed recently [11]. We also

want to take into account scale-free properties by investigating our model on the network of Barabási and Albert [12, 13]. Among others, already the Hopfield model [14, 15] and the Hindmarsh-Rose neural model [16] have been studied before on such networks.

In order to describe the neural activity we use a variant of the original model of McCulloch and Pitts [17]. A neuron i can be in two states, firing or non-firing, described by binary variables $x_i = 1$ (active) or $x_i = 0$ (inactive). They are initialized randomly. The state of a neuron is updated in time t through the following equation [18]:

$$x_i(t) = \Theta\Big(\sum_{j \in (i)}^{n} S_{ij} x_j(t-1) + T \Big), \qquad (1)$$

where n is the number of connections of each neuron and S_{ij} represents the strength of the synapse between neurons i and j. The strength factors S_{ij} are set randomly to be either $+1$ or -1 with equal probability, representing either an excitatory or an inhibitory neuron, respectively. The variable T is the firing threshold which in fact throughout this paper is taken to be zero. Here, Θ denotes the Heaviside function defined as usual: $\Theta(z) = 1$ if $z \geq 0$, and $\Theta(z) = 0$ if $z < 0$.

We are interested in analyzing the overall firing activity of the brain which in practice can be monitored, for instance, through EEG measurements. For that we define a macroscopic variable of our model which we call the "neural activity" $A(t)$ as the fraction of neurons firing at time t,

$$A(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t). \qquad (2)$$

The small-world network of Watts and Strogatz [1, 2, 3] allows one to continuously connect two extreme situations, namely the *regular lattice* and *random graph* of Renyi and Erdös [19]. The construction of a Watts-Strogatz network is performed in two steps:

1. We start with a regular lattice, in our case a one-dimensional chain of N sites with connections between nearest and next-nearest neighbors and periodic boundary conditions so that the total number of connections per site is $k = 4$.
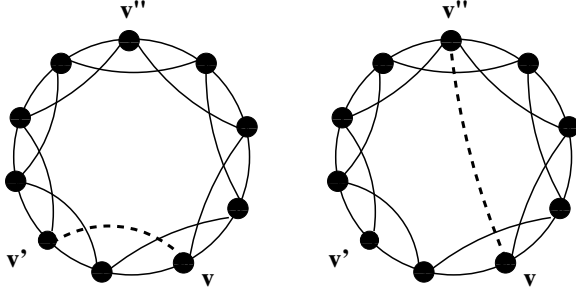
FIG. 1: One starts with a regular chain having nearest and next-nearest neighbor connections. With probability p we replace short-range by long-range connections by rewiring: Initially v is connected to v'. But after rewiring, the connection of v to v' is replaced by another connection, say v to v''.

2. With probability p ("rewiring probability") we replace for each site i a connection L_{ij} by another one L_{ix} where x is any randomly chosen site (Fig. 1).

The rewiring probability p varies between zero and one and is the main parameter of our investigation. For $p = 0$ we have a regular lattice and for $p = 1$ a random graph.

The scale-free network of Barabási and Albert [12, 13] is constructed by starting with a small number m_0 of nodes at time $t = 0$. Then, at each time step one adds a new node having $m < m_0$ links to the existing nodes. The probability that a new node is connected to node i is $k_i / \sum_j k_j$ where k_i is the actual connectivity of node i.

This rule assures preferential attachment to sites of higher connectivity. As a result the distribution of connectivities also called the "degree distribution" follows a power-law $P(k) \sim k^{-\gamma}$ with an exponent $\gamma = 3$. This property characterizes a scale-free network.

We first consider a network with $N = 2048$ sites and modify the rewiring probability p between zero and one. For each value of p, we generate 1000 different networks and compute at each time step the neural activity $A(t)$, Eq. (2), as generated by the model of Eq. (1) Figure 2a shows the non-periodic (chaotic) temporal evolution of the neural activity as obtained for a Watts-Strogatz network with rewiring probability $p = 0$ (i.e. on a regular chain). In order to measure the periodicity in time, we analyze the last 1024 time steps of series with 16382 steps using a shift algorithm In this way, it is possible to detect all periods shorter than 512 time steps.

Figure 2b shows the result when the rewiring probability is $p = 1$ (i.e. on a random graph): the time-series becomes periodic. To quantify the degree of periodicity ϕ, i.e., the fraction of graphs that exhibit periodic time series, we performed additional simulations on networks with $N = 1024, 2048, 4096, 8192$ and 16384 sites. For each size we generated 1000 different networks and measured the degree of periodicity for various values of p.

The inset of Figure 3 shows the fraction ϕ of networks reaching a periodic regime as a function of the rewiring probability p. For $p = 0$ we find consistently $\phi = 0$, i.e. all
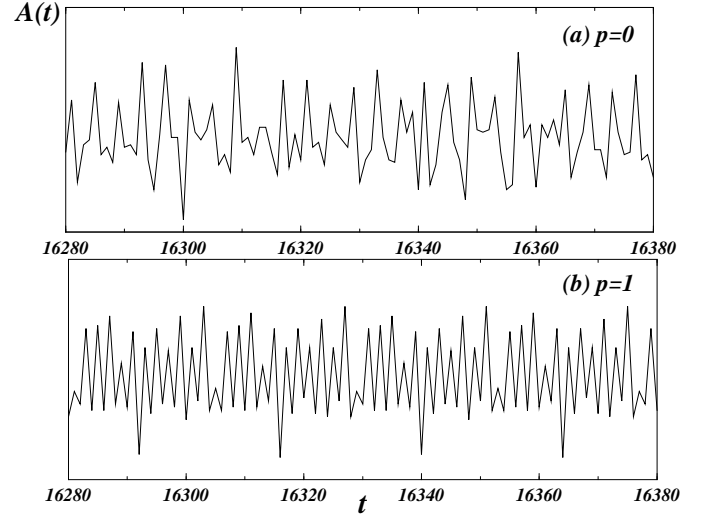

FIG. 2: Temporal evolution of the neural activity of graphs having 2048 sites. (a) for a rewiring probability $p = 0$, i.e. a regular network. (b) for a rewiring probability $p = 1$, i.e. a random graph.
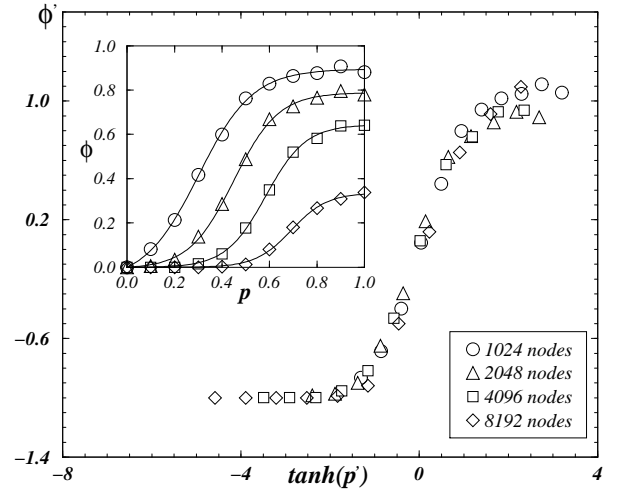

FIG. 3: Renormalized fraction $\phi'(N)$ of graphs with periodic time series as a function of the tanh of the renormalized rewiring probability $p'(N)$. Inset: Fraction ϕ of graphs with periodic time series as a function of the rewiring probability for different network sizes N.

networks behave chaotically regardless of their size. Furthermore, the solid lines clearly indicate that the increase of the fraction ϕ with p can be closely described by the expression,

$$\phi(p, N) = a_0 \Big[\tanh(\frac{p}{a_1} + a_2) - \tanh(a_2) \Big], \qquad (3)$$

where the parameters a_0, a_1 and a_2 are obtained through the best nonlinear fit to the data of Eq. (3 for each different value of N. In addition, our results suggest that the parameters a_i depend only on the system size N, since we find their behav-
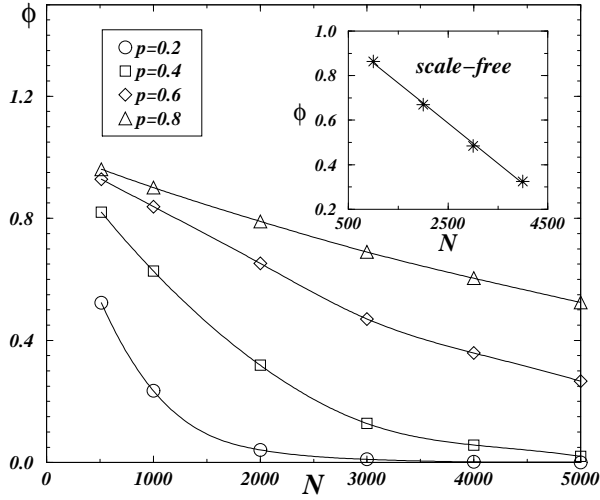
FIG. 4: Fraction ϕ of graphs with periodic time series as a function of the size N of the graph for different values of p. The inset shows the data for the Barabási-Albert network for $m = 3$.

FIG. 5: Double-logarithmic plot of the average period length for the Watts-Strogatz network as function of the size N of the network averaged over 1000 networks for $p = 0.9$. The inset shows the data for the Barabási-Albert network for $m = 3$.

iors to be well represented by the relations,

$$a_0(N) = \alpha_0 + \beta_0 N \tag{4}$$

$$a_1(N) = \alpha_1 + \beta_1 \ln N \tag{5}$$

$$a_2(N) = \alpha_2 + \beta_2 \ln N \tag{6}$$

with $\alpha_0 = 0.501$, $\alpha_1 = 0.476$, $\alpha_2 = 9.610$, $\beta_0 = -4.146 \times 10^{-5}$, $\beta_1 = -0.036$, and $\beta_2 = -1.576$. Following this approach, we can rescale the variables ϕ and p as $\phi' = \phi/a_0 + \tanh(a_2)$ and $p' = p/a_1 + a_2$, respectively, to show that all data can be collapsed on the top of each other as displayed in Fig. 3. From the inset in Fig. 3 we can also see that the degree of periodicity ϕ decays with increasing size N of the network for a fixed value of $p > 0$.

As can be seen from Fig. 3, there exists a $N_0 \approx 32,000$ defined through $\phi'(N_0) = 0$ above which the fraction ϕ of graphs with periodic signals is zero, N_0 being the largest network size still showing periods.

In Fig. 4 we show that the degree of periodicity ϕ decays with the increase of the network size N, and that this effect becomes more pronounced the smaller is the value of p.

The length of a period is the minimum number of time steps for which the time series is repeated. Figure 5 shows in double-logarithmic scale the average period length as a function of the size N of the network for $N = 128, 256, 512, 1024$ and 2048 for a fixed value of $p = 0.9$. For each value of N, we average over 1000 networks from series that have a total length of 16384 time steps, the last 1024 of which were analyzed. We see that the average period $\langle T \rangle$ increases with the size like a power law, with an exponent that is approximately equal to $1/2$, i.e., $\langle T \rangle \propto \sqrt{N}$.
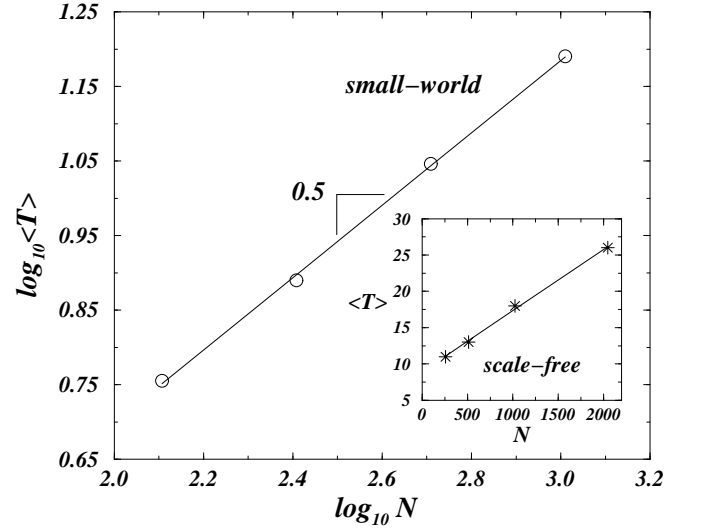
In the insets of Figs. 4 and 5 we see the corresponding data for a Barabási-Albert network with $m = 3$. Here the fraction of graphs having periodic signals decreases with the system size N linearly and at around $N_0 \approx 6000$ it becomes zero so that for larger sizes no periods can be found. The average period length increases linearly with the size: $\langle T \rangle \propto N$.

In conclusion, we have shown that a simple neural model on Watts-Strogatz and Barabási-Albert networks can generate periodic activity signals, however, only if the networks are not too large. These cycles become more frequent if one has more long-range connections, and the length of their period increases like the number of neurons or its square root for scale-free or small-world networks respectively. Since the brain is huge one would therefore expect the present mechanism to be only relevant in very small fractions of the brain. In particular one can imagine that our discovery is important to explain periodic signals for instance in pace makers, nervous systems of lower animals or other very small units of neurons.

The brain is one of the big challenges of our century and still full of mysteries and contradictions. From EEG and direct measurements with electrodes we know that cycles of firing do appear. Their role and their origin are still not clear. Of course they are only finite in length and a perfect periodicity has also not yet been confirmed. Our finding proposes at least one mechanism that might explain their origin. We see that on small scales the network complexity, be it scale-free or small-world can spontaneously generate periodic signals. We have evidence that real neural networks do have both aspects. In particular small-world topology has been experimentally evidenced. Small units of neurons could be the nucleus that generates periodic signals using the mechanism found in our work. How they interact with the rest of the brain is not clear and it would be interesting to study this in more detail. We

also propose to investigate the spatial distribution of periodic signals to ask the question if they could be localized in space or if they are collective excitations. Larger systems should be studied investing more CPU time. Another aspect we will study in the future is the effect of pinning. What happens when one neuron has an externally controlled fixed signal (e.g. firing all the time). Can we suppress or induce oscillations in this way? As they stand our results concern small units of neurons and an experimental verification could be imagined with brains of low animals or with in vitro cultures. One direct application could be the pace-maker of the heart which is indeed a neural net providing a perfectly periodic signal over very long times.

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